CHAPTER 1

Oscillatory Motion

The study of vibration is concerned with the oscillatory motions of bodies and the forces associated with them. All bodies possessing mass and elasticity are capable of vibration. Thus, most engineering machines and structures experience vibration to some degree, and their design generally requires consideration of their oscillatory behavior.

Oscillatory systems can be broadly characterized as *linear* or *nonlinear*. For linear systems, the principle of superposition holds, and the mathematical techniques available for their treatment are well developed. In contrast, techniques for the analysis of nonlinear systems are less well known, and difficult to apply. However, some knowledge of nonlinear systems is desirable, because all systems tend to become nonlinear with increasing amplitude of oscillation.

There are two general classes of vibrations—free and forced. *Free vibration* takes place when a system oscillates under the action of forces inherent in the system itself, and when external impressed forces are absent. The system under free vibration will vibrate at one or more of its *natural frequencies,* which are properties of the dynamical system established by its mass and stiffness distribution.

Vibration that takes place under the excitation of external forces is called *forced vibration.* When the excitation is oscillatory, the system is forced to vibrate at the excitation frequency. If the frequency of excitation coincides with one of the natural frequencies of the system, a condition of *resonance* is encountered, and dangerously large oscillations may result. The failure of major structures such as bridges, buildings, or airplane wings is an awesome possibility under resonance. Thus, the calculation of the natural frequencies is of major importance in the study of vibrations.

Vibrating systems are all subject to *damping* to some degree because energy is dissipated by friction and other resistances. If the damping is small, it has very little influence on the natural frequencies of the system, and hence the calculations for the natural frequencies are generally made on the basis of no damping. On the other hand, damping is of great importance in limiting the amplitude of oscillation at resonance.

The number of independent coordinates required to describe the motion of a system is called *degrees of freedom* of the system. Thus, a free particle undergoing general motion in space will have three degrees of freedom, and a rigid body will have six

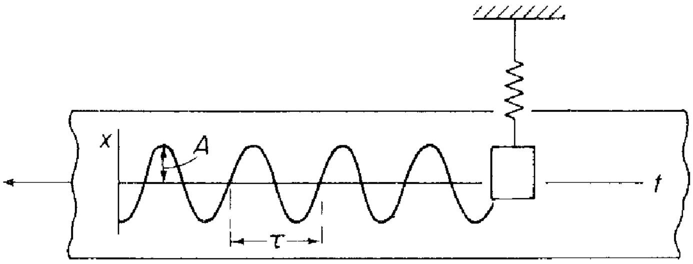

FIGURE 1.1.1. Recording harmonic motion.

degrees of freedom, i.e., three components of position and three angles defining its orientation. Furthermore, a continuous elastic body will require an infinite number of coordinates (three for each point on the body) to describe its motion; hence, its degrees of freedom must be infinite. However, in many cases, parts of such bodies may be assumed to be rigid, and the system may be considered to be dynamically equivalent to one having finite degrees of freedom. In fact, a surprisingly large number of vibration problems can be treated with sufficient accuracy by reducing the system to one having a few degrees of freedom.

1.1 HARMONIC MOTION

Oscillatory motion may repeat itself regularly, as in the balance wheel of a watch, or display considerable irregularity, as in earthquakes. When the motion is repeated in equal intervals of time τ, it is called *periodic motion*. The repetition time τ is called the *period* of the oscillation, and its reciprocal, $f = 1/\tau$, is called the *frequency*. If the motion is designated by the time function $x(t)$, then any periodic motion must satisfy the relationship $x(t) = x(t + \tau)$.

The simplest form of periodic motion is *harmonic motion*. It can be demonstrated by a mass suspended from a light spring, as shown in Fig. 1.1.1. If the mass is displaced from its rest position and released, it will oscillate up and down. By placing a light source on the oscillating mass, its motion can be recorded on a light-sensitive filmstrip, which is made to move past it at a constant speed.

The motion recorded on the filmstrip can be expressed by the equation

$$x = A \sin 2\pi \frac{t}{\tau} \tag{1.1.1}$$

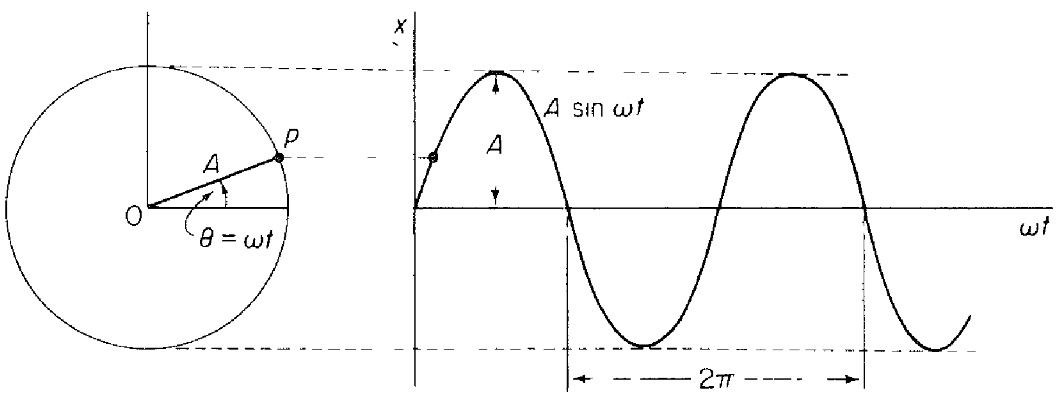

FIGURE 1.1.2. Harmonic motion as a projection of a point moving on a circle.

where A is the amplitude of oscillation, measured from the equilibrium position of the mass, and τ is the period. The motion is repeated when $t = \tau$.

Harmonic motion is often represented as the projection on a straight line of a point that is moving on a circle at constant speed, as shown in Fig. 1.1.2. With the angular speed of the line $0-p$ designated by ω, the displacement x can be written as

$$x = A \sin \omega t \tag{1.1.2}$$

The quantity ω is generally measured in radians per second, and is referred to as the *circular frequency*.[1] Because the motion repeats itself in 2π radians, we have the relationship

$$\omega = \frac{2\pi}{\tau} = 2\pi f \tag{1.1.3}$$

where τ and f are the period and frequency of the harmonic motion, usually measured in seconds and cycles per second, respectively.

The velocity and acceleration of harmonic motion can be simply determined by differentiation of Eq. (1.1.2). Using the dot notation for the derivative, we obtain

$$\dot{x} = \omega A \cos \omega t = \omega A \sin (\omega t + \pi/2) \tag{1.1.4}$$

$$\ddot{x} = -\omega^2 A \sin \omega t = \omega^2 A \sin (\omega t + \pi) \tag{1.1.5}$$

Thus, the velocity and acceleration are also harmonic with the same frequency of oscillation, but lead the displacement by $\pi/2$ and π radians, respectively. Figure 1.1.3 shows both time variation and the vector phase relationship between the displacement, velocity, and acceleration in harmonic motion.

Examination of Eqs. (1.1.2) and (1.1.5) reveals that

$$\ddot{x} = -\omega^2 x \tag{1.1.6}$$

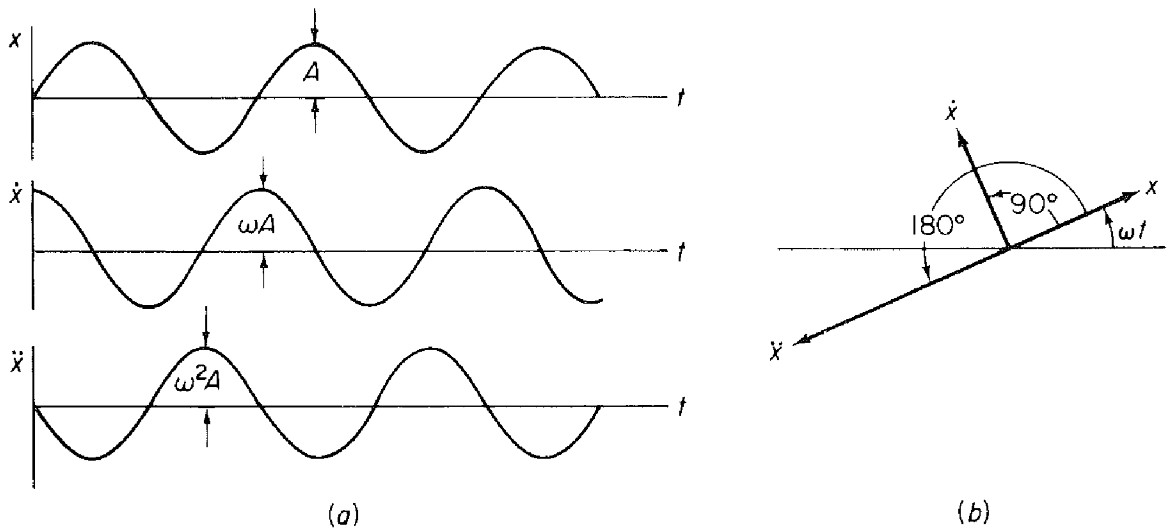

FIGURE 1.1.3. In harmonic motion, the velocity and acceleration lead the displacement by $\pi/2$ and π.

[1]The word *circular* is generally deleted, and ω and f are used without distinction for frequency.

so that in harmonic motion, the acceleration is proportional to the displacement and is directed toward the origin. Because Newton's second law of motion states that the acceleration is proportional to the force, harmonic motion can be expected for systems with linear springs with force varying as kx.

Exponential form. The trigonometric functions of sine and cosine are related to the exponential function by Euler's equation

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.1.7}$$

A vector of amplitude A rotating at constant angular speed ω can be represented as a complex quantity z in the Argand diagram, as shown in Fig. 1.1.4.

$$\begin{aligned} z &= Ae^{i\omega t} \\ &= A\cos\omega t + iA\sin\omega t \\ &= x + iy \end{aligned} \tag{1.1.8}$$

The quantity z is referred to as the *complex sinusoid*, with x and y as the real and imaginary components, respectively. The quantity $z = Ae^{i\omega t}$ also satisfies the differential equation (1.1.6) for harmonic motion.

Figure 1.1.5 shows z and its conjugate $z^* = Ae^{-i\omega t}$, which is rotating in the negative direction with angular speed $-\omega$. It is evident from this diagram that the real component x is expressible in terms of z and z^* by the equation

$$x = \tfrac{1}{2}(z + z^*) = A\cos\omega t = \operatorname{Re} Ae^{i\omega t} \tag{1.1.9}$$

where Re stands for the real part of the quantity z. We will find that the exponential form of the harmonic motion often offers mathematical advantages over the trigonometric form.

Some of the rules of exponential operations between $z_1 = A_1 e^{i\theta_1}$ and $z_2 = A_2 e^{i\theta_2}$ are as follows:

$$\begin{aligned} &\text{Multiplication} \qquad z_1 z_2 = A_1 A_2 e^{i(\theta_1+\theta_2)} \\ &\text{Division} \qquad \frac{z_1}{z_2} = \left(\frac{A_1}{A_2}\right) e^{i(\theta_1-\theta_2)} \end{aligned} \tag{1.1.10}$$

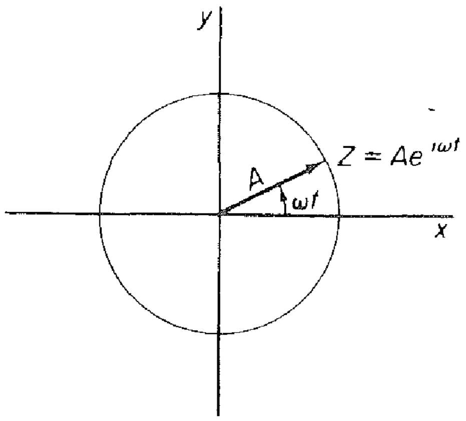

FIGURE 1.1.4. Harmonic motion represented by a rotating vector.

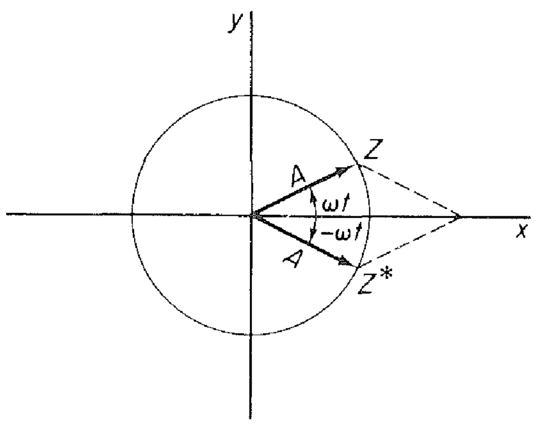

FIGURE 1.1.5. Vector z and its conjugate z^*.

Powers
$$z^n = A^n e^{in\theta}$$
$$z^{1/n} = A^{1/n} e^{i\theta/n}$$

1.2 PERIODIC MOTION

It is quite common for vibrations of several different frequencies to exist simultaneously. For example, the vibration of a violin string is composed of the fundamental frequency f and all its harmonics, $2f$, $3f$, and so forth. Another example is the free vibration of a multidegree-of-freedom system, to which the vibrations at each natural frequency contribute. Such vibrations result in a complex waveform, which is repeated periodically as shown in Fig. 1.2.1.

The French mathematician J. Fourier (1768–1830) showed that any periodic motion can be represented by a series of sines and cosines that are harmonically related. If $x(t)$ is a periodic function of the period τ, it is represented by the Fourier series

$$x(t) = \frac{a_0}{2} + a_1 \cos \omega_1 t + a_2 \cos \omega_2 t + \cdots + b_1 \sin \omega_1 t + b_2 \sin \omega_2 t + \cdots \tag{1.2.1}$$

where

$$\omega_1 = \frac{2\pi}{\tau}$$
$$\omega_n = n\omega_1$$

To determine the coefficients a_n and b_n, we multiply both sides of Eq. (1.2.1) by $\cos \omega_n t$ or $\sin \omega_n t$ and integrate each term over the period τ. By recognizing the following relations,

$$\int_{-\tau/2}^{\tau/2} \cos \omega_n t \cos \omega_m t \, dt = \begin{cases} 0 & \text{if } m \neq n \\ \tau/2 & \text{if } m = n \end{cases}$$
$$\int_{-\tau/2}^{\tau/2} \sin \omega_n t \sin \omega_m t \, dt = \begin{cases} 0 & \text{if } m \neq n \\ \tau/2 & \text{if } m = n \end{cases} \tag{1.2.2}$$

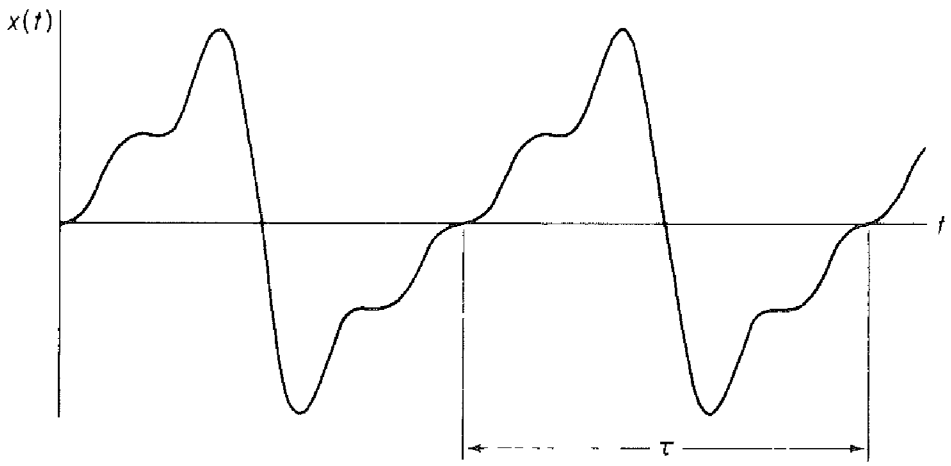

FIGURE 1.2.1. Periodic motion of period τ.

$$\int_{-\tau/2}^{\tau/2} \cos \omega_n t \sin \omega_m t \, dt = \begin{cases} 0 & \text{if } m \neq n \\ 0 & \text{if } m = n \end{cases}$$

all terms except one on the right side of the equation will be zero, and we obtain the result

$$a_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} x(t) \cos \omega_n t \, dt$$

$$b_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} x(t) \sin \omega_n t \, dt \tag{1.2.3}$$

The Fourier series can also be represented in terms of the exponential function. Substituting

$$\cos \omega_n t = \tfrac{1}{2}(e^{i\omega_n t} + e^{-i\omega_n t})$$

$$\sin \omega_n t = -\tfrac{1}{2}i(e^{i\omega_n t} - e^{-i\omega_n t})$$

in Eq. (1.2.1), we obtain

$$\begin{aligned} x(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\tfrac{1}{2}(a_n - ib_n)e^{i\omega_n t} + \tfrac{1}{2}(a_n + ib_n)e^{-i\omega_n t}\right] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[c_n e^{i\omega_n t} + c_n^* e^{-i\omega_n t}\right] \\ &= \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \end{aligned} \tag{1.2.4}$$

where

$$\begin{aligned} c_0 &= \tfrac{1}{2}a_0 \\ c_n &= \tfrac{1}{2}(a_n - ib_n) \end{aligned} \tag{1.2.5}$$

Substituting for a_n and b_n from Eq. (1.2.3), we find c_n to be

$$\begin{aligned} c_n &= \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} x(t)(\cos \omega_n t - i \sin \omega_n t) \, dt \\ &= \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} x(t) e^{-i\omega_n t} \, dt \end{aligned} \tag{1.2.6}$$

Some computational effort can be minimized when the function $x(t)$ is recognizable in terms of the even and odd functions:

$$x(t) = E(t) + O(t) \tag{1.2.7}$$

An even function $E(t)$ is symmetric about the origin, so that $E(t) = E(-t)$, i.e., $\cos \omega t = \cos(-\omega t)$. An odd function satisfies the relationship $O(t) = -O(-t)$, i.e., $\sin \omega t = -\sin(-\omega t)$. The following integrals are then helpful:

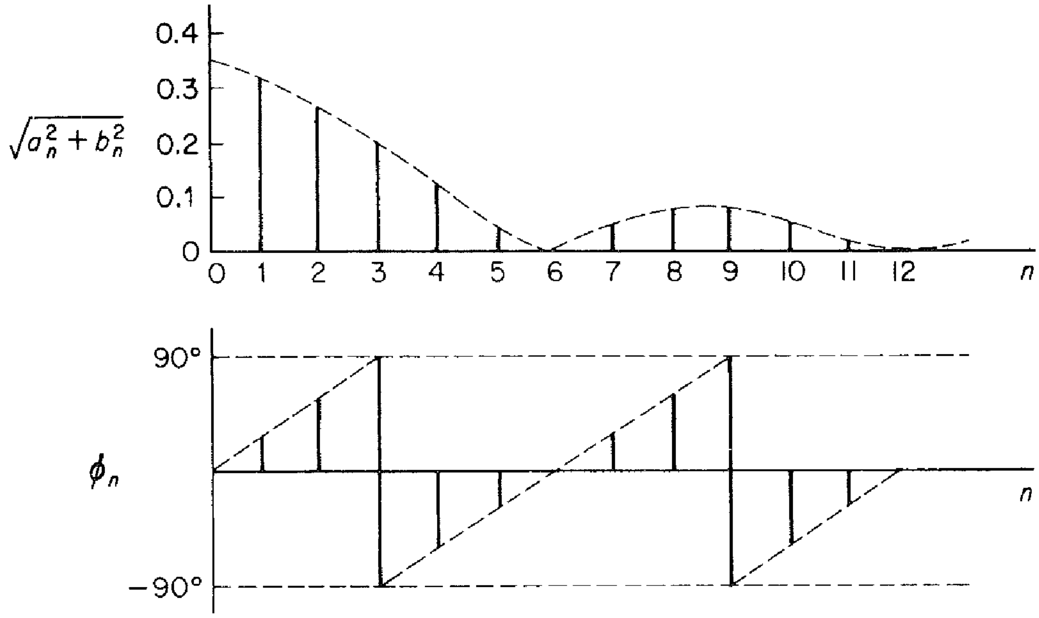

FIGURE 1.2.2. Fourier spectrum for pulses shown in Prob. 1.16, $k = \frac{1}{3}$.

$$\int_{-\tau/2}^{\tau/2} E(t) \sin \omega_n t \, dt = 0 \tag{1.2.8}$$

$$\int_{-\tau/2}^{\tau/2} O(t) \cos \omega_n t \, dt = 0$$

When the coefficients of the Fourier series are plotted against frequency ω_n, the result is a series of discrete lines called the *Fourier spectrum.* Generally plotted are the absolute values $|2c_n| = \sqrt{a_n^2 + b_n^2}$ and the phase $\phi_n = \tan^{-1}(b_n/a_n)$, an example of which is shown in Fig. 1.2.2. Fourier analysis including the Fourier transform are discussed in more detail in Chapter 13.

With the aid of the digital computer, harmonic analysis today is efficiently carried out. A computer algorithm known as the *fast Fourier transform*[2] (FFT) is commonly used to minimize the computation time.

1.3 VIBRATION TERMINOLOGY

Certain terminologies used in vibration analysis need to be represented here. The simplest of these are the *peak value* and the *average value.*

The peak value generally indicates the maximum stress that the vibrating part is undergoing. It also places a limitation on the "rattle space" requirement.

The average value indicates a steady or static value, somewhat like the dc level of an electrical current. It can be found by the time integral

$$\bar{x} = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t) \, dt \tag{1.3.1}$$

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[2]See J. S. Bendat and A. G. Piersol, *Random Data* (New York: John Wiley, 1971), pp. 305–306.

For example, the average value for a complete cycle of a sine wave, $A \sin t$, is zero; whereas its average value for a half-cycle is

$$\bar{x} = \frac{A}{\pi} \int_0^{\pi} \sin t \, dt = \frac{2A}{\pi} = 0.637A$$

It is evident that this is also the average value of the rectified sine wave shown in Fig. 1.3.1.

The square of the displacement generally is associated with the energy of the vibration for which the mean square value is a measure. The *mean square value* of a time function $x(t)$ is found from the average of the squared values, integrated over some time interval T:

$$\overline{x^2} = \lim_{T \to \infty} \frac{1}{T} \int_0^T x^2(t) \, dt \tag{1.3.2}$$

For example, if $x(t) = A \sin \omega t$, its mean square value is

$$\overline{x^2} = \lim_{T \to \infty} \frac{A^2}{T} \int_0^T \frac{1}{2}(1 - \cos 2\omega t) \, dt = \frac{1}{2} A^2$$

The *root mean square* (rms) value is the square root of the mean square value. From the previous example, the rms of the sine wave of amplitude A is $A/\sqrt{2} = 0.707A$. Vibrations are commonly measured by rms meters.

The *decibel* is a unit of measurement that is frequently used in vibration measurements. It is defined in terms of a power ratio.

$$\begin{aligned} \text{dB} &= 10 \log_{10}\left(\frac{p_1}{p_2}\right) \\ &= 10 \log_{10}\left(\frac{x_1}{x_2}\right)^2 \end{aligned} \tag{1.3.3}$$

The second equation results from the fact that power is proportional to the square of the amplitude or voltage. The decibel is often expressed in terms of the first power of amplitude or voltage as

$$\text{dB} = 20 \log_{10}\left(\frac{x_1}{x_2}\right) \tag{1.3.4}$$

Thus an amplifier with a voltage gain of 5 has a decibel gain of

$$20 \log_{10}(5) = +14$$

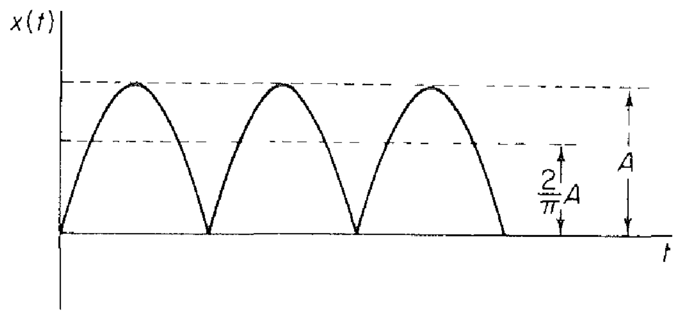

FIGURE 1.3.1. Average value of a rectified sine wave.

Because the decibel is a logarithmic unit, it compresses or expands the scale.

When the upper limit of a frequency range is twice its lower limit, the frequency span is said to be an *octave*. For example, each of the frequency bands in the following table represents an octave band.

Band	Frequency Range (Hz)	Frequency Bandwidth
1	10–20	10
2	20–40	20
3	40–80	40
4	200–400	200

PROBLEMS

1.1. A harmonic motion has an amplitude of 0.20 cm and a period of 0.15 s. Determine the maximum velocity and acceleration.

1.2. An accelerometer indicates that a structure is vibrating harmonically at 82 cps with a maximum acceleration of 50 g. Determine the amplitude of vibration.

1.3. A harmonic motion has a frequency of 10 cps and its maximum velocity is 4.57 m/s. Determine its amplitude, its period, and its maximum acceleration.

1.4. Find the sum of two harmonic motions of equal amplitude but of slightly different frequencies. Discuss the beating phenomena that result from this sum.

1.5. Express the complex vector $4 + 3i$ in the exponential form $Ae^{i\theta}$.

1.6. Add two complex vectors $(2 + 3i)$ and $(4 - i)$, expressing the result as $A\angle\theta$.

1.7. Show that the multiplication of a vector $z = Ae^{i\omega t}$ by i rotates it by 90°.

1.8. Determine the sum of two vectors $5e^{i\pi/6}$ and $4e^{i\pi/3}$ and find the angle between the resultant and the first vector.

1.9. Determine the Fourier series for the rectangular wave shown in Fig. P1.9.

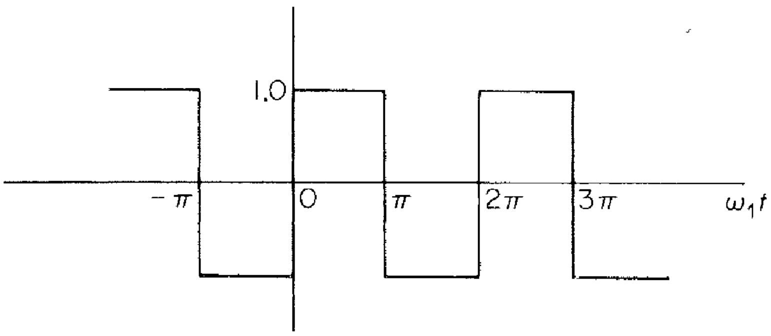

FIGURE P1.9.

1.10. If the origin of the square wave of Prob. 1.9 is shifted to the right by $\pi/2$, determine the Fourier series.

1.11. Determine the Fourier series for the triangular wave shown in Fig. P1.11.

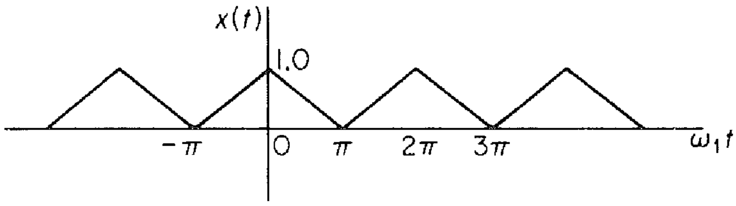

FIGURE P1.11.

1.12. Determine the Fourier series for the sawtooth curve shown in Fig. P1.12. Express the result of Prob. 1.12 in the exponential form of Eq. (1.2.4).

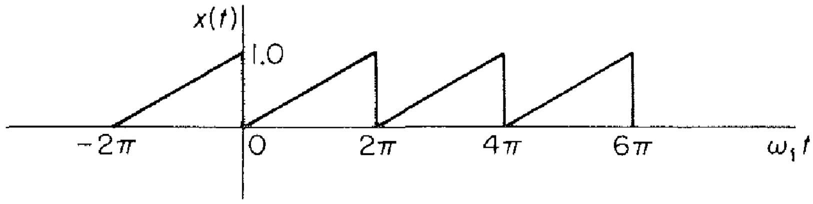

FIGURE P1.12.

1.13. Determine the rms value of a wave consisting of the positive portions of a sine wave.

1.14. Determine the mean square value of the sawtooth wave of Prob. 1.12. Do this two ways, from the squared curve and from the Fourier series.

1.15. Plot the frequency spectrum for the triangular wave of Prob. 1.11.

1.16. Determine the Fourier series of a series of rectangular pulses shown in Fig. P1.16. Plot c_n and ϕ_n versus n when $k = \frac{2}{3}$.

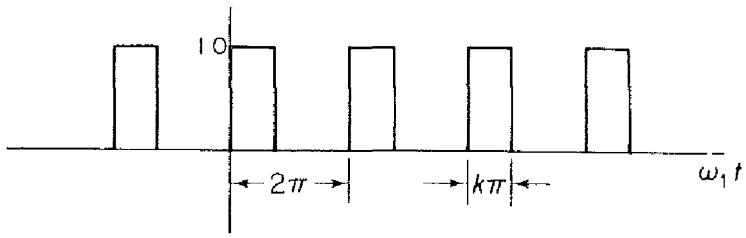

FIGURE P1.16.

1.17. Write the equation for the displacement s of the piston in the crank-piston mechanism shown in Fig. P1.17, and determine the harmonic components and their relative magnitudes. If $r/l = \frac{1}{3}$, what is the ratio of the second harmonic compared to the first?

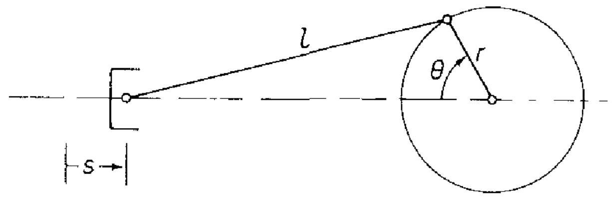

FIGURE P1.17.

1.18. Determine the mean square of the rectangular pulse shown in Fig. P1.18 for $k = 0.10$. If the amplitude is A, what would an rms voltmeter read?

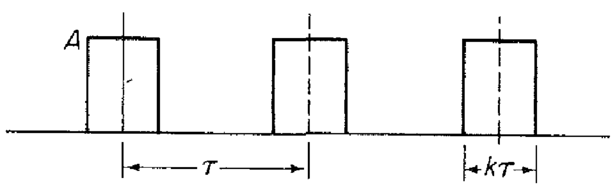

FIGURE P1.18.

1.19. Determine the mean square value of the triangular wave of Fig. P1.11.

1.20. An rms voltmeter specifies an accuracy of ± 0.5 dB. If a vibration of 2.5 mm rms is measured, determine the millimeter accuracy as read by the voltmeter.

1.21. Amplification factors on a voltmeter used to measure the vibration output from an accelerometer are given as 10, 50, and 100. What are the decibel steps?

1.22. The calibration curve of a piezoelectric accelerometer is shown in Fig. P1.22 where the ordinate is in decibels. If the peak is 32 dB, what is the ratio of the resonance response to that at some low frequency, say, 1000 cps?

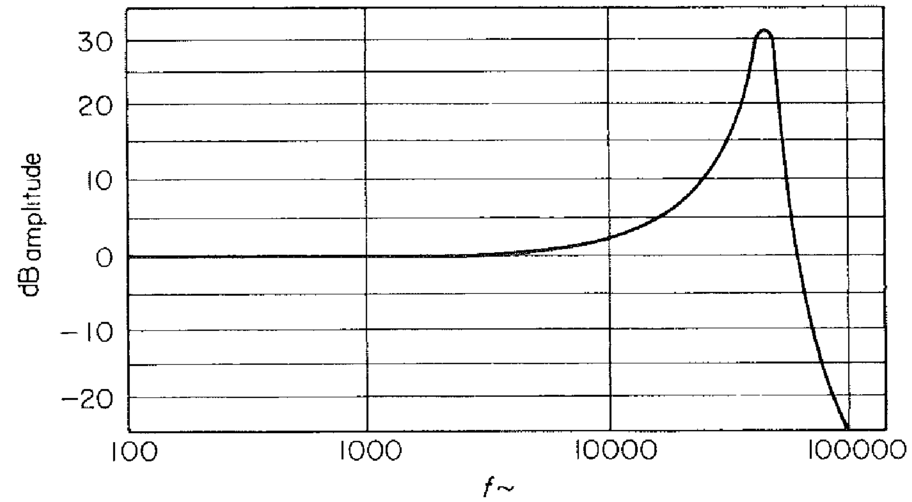

FIGURE P1.22.

1.23. Using coordinate paper similar to that of Appendix A, outline the bounds for the following vibration specifications. Max. acceleration = 2 g, max. displacement = 0.08 in., min. and max. frequencies: 1 Hz and 200 Hz.

1.24. Assume a pulse occurs at integer times and lasts for 1 second. It has a random amplitude with the probability of having the amplitude equal 1 or -1 being $p(1) = p(-1) = 1/2$. What is the mean value and the mean square value of the amplitude?

1.25. Show that every function $f(t)$ can be represented as a sum of an odd function $O(t)$ and an even function $E(t)$.